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Engineering Notes

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On the Uniqueness of the Minimum-State Representation

E. Nissim*
Technion—Israel Institute of Technology, 32000 Haifa, Israel

Introduction

THE minimum-state rational function approximation[1] (MIST RFA) is a well-established approximation used in flutter and in aeroservoelasticity work. The MIST approximation leads to reduced-order equations and thus is very attractive also for multidisciplinary optimization (MDO). When applying MDO to aircraft design problems that involve configuration changes, gradients of the coefficients in the MIST representation need to be evaluated. The nonuniqueness of the very important [D] and [E] matrices in the MIST representation makes the computation of the gradients of these latter matrices meaningless with respect to any design variable. The problems associated with the nonuniqueness of the [D] and [E] matrices are discussed by E. Livne in in a recent AIAA paper.[2] Once the [D] and [E] matrices are made unique, this obstacle is removed and enables the use of MIST over a wider range of problems.

Mathematical Representation

Let matrix $\boldsymbol{A}$ represent the aerodynamic generalized matrix, and let the MIST representation of $\boldsymbol{A}$ be given by

$$\boldsymbol{A} = \boldsymbol{A}_0 + ik\boldsymbol{A}_1 - k^2\boldsymbol{A}_2 + ik\boldsymbol{D}\gamma\boldsymbol{E} \tag{1}$$

where $\boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{D}$, and $\boldsymbol{E}$ are all real matrices and γ is a diagonal matrix defined by

$$\gamma = \begin{bmatrix} \dfrac{1}{ik+\gamma_1} & & & \\ & \dfrac{1}{ik+\gamma_2} & & \\ & & \ddots & \\ & & & \dfrac{1}{ik+\gamma_l} \end{bmatrix} \tag{2}$$

where $i = \sqrt{(-1)}$ and k is the reduced frequency. Matrix γ will be referred to as the lags matrix (of order $l \times l$). Let $\boldsymbol{A}$ be of order $n \times m$. Matrices $\boldsymbol{D}$ and $\boldsymbol{E}$ will be of order $n \times l$ and $l \times m$, respectively.

It is easy to show that irrespective of the way matrices $\boldsymbol{D}$ and $\boldsymbol{E}$ are determined, they are not unique. For instance, one can multiply matrix $\boldsymbol{D}$ by scalar c and divide matrix $\boldsymbol{E}$ by the very same scalar c, thus leaving the combined $\boldsymbol{D}\gamma\boldsymbol{E}$ term unaffected. Unique determination of matrices $\boldsymbol{D}$ and $\boldsymbol{E}$ is not always required. However, in some instances, such as those just stated, unique values of $\boldsymbol{D}$ and $\boldsymbol{E}$ are essential. Nonunique solutions are known to exist in different fields of mathematics. Eigenvectors for example are made unique by normalizing the length of each eigenvector solution. The singular value decomposition of a matrix is made unique by imposing some orthogonality conditions on the resulting matrices. In the following, an attempt will be made to cast matrices $\boldsymbol{D}$ and $\boldsymbol{E}$ into a unique form, without affecting the numerical values of the product $\boldsymbol{D}\gamma\boldsymbol{E}$ term, which appears in Eq. (1).

Uniqueness Considerations

It is obvious that the combined $\boldsymbol{D}\gamma\boldsymbol{E}$ term will remain unaffected when the following general matrix operations are performed:

$$\boldsymbol{D}\gamma\boldsymbol{E} = \boldsymbol{D}\boldsymbol{G}\boldsymbol{G}^{-1}\gamma\boldsymbol{E} \tag{3}$$

where $\boldsymbol{G}$ is any nonsingular matrix of order $l \times l$. A question now arises regarding the possibility of choosing a matrix $\boldsymbol{G}$ in such a way as to yield a new $\boldsymbol{D}\gamma\boldsymbol{E}$ term in Eq. (1) such that the new $\boldsymbol{D}$ and $\boldsymbol{E}$ matrices become unique.

Before proceeding to determine a possible $\boldsymbol{G}$ matrix, it is convenient to bring Eq. (3) into the following form:

$$\begin{aligned} \boldsymbol{D}\gamma\boldsymbol{E} &= \boldsymbol{D}\boldsymbol{G}\boldsymbol{G}^{-1}\gamma\boldsymbol{E} \\ &= \boldsymbol{D}\boldsymbol{G}\gamma\gamma^{-1}\boldsymbol{G}^{-1}\gamma\boldsymbol{E} \end{aligned} \tag{4}$$

If we now define

$$\bar{\boldsymbol{D}} = \boldsymbol{D}\boldsymbol{G}$$

and

$$\bar{\boldsymbol{E}} = \gamma^{-1}\boldsymbol{G}^{-1}\gamma\boldsymbol{E} \tag{5}$$

Then clearly

$$\boldsymbol{D}\gamma\boldsymbol{E} = \bar{\boldsymbol{D}}\gamma\bar{\boldsymbol{E}} \tag{6}$$

It is now required to find a $\boldsymbol{G}$ matrix that will turn $\bar{\boldsymbol{D}}$ unique, and thus also lead to a unique $\bar{\boldsymbol{E}}$ matrix.

From Eq. (5) it follows that $\bar{\boldsymbol{E}}$ will in general assume complex values, except for the trivial case where all γ_l values are equal. This is true because the diagonal matrix γ is complex [see Eq. (2)]. Because $\bar{\boldsymbol{E}}$ *must* maintain real values and γ must maintain a general nontrivial form, one is led to the conclusion that matrix $\bar{\boldsymbol{E}}$ can be made real *only* if matrix $\boldsymbol{G}$ assumes a diagonal form. In this latter case, the term $\gamma^{-1}\boldsymbol{G}^{-1}\gamma$ in Eq. (5) simplifies to $\boldsymbol{G}^{-1}$, thus yielding

$$\bar{\boldsymbol{E}} = \boldsymbol{G}^{-1}\boldsymbol{E} \tag{7}$$

where, as already stated, $\boldsymbol{G}$ *must* be a diagonal matrix.

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*Professor Emeritus, Faculty of Aerospace Engineering; currently Visting Scholar, University of California, Los Angeles, 90095-1597.

Determination of Matrix *G*

The requirement that $\boldsymbol{G}$ be diagonal indicates that a *sufficient* condition for the uniqueness of matrix $\bar{\boldsymbol{D}}$ is the simple requirement that the length of each column of $\bar{\boldsymbol{D}}$ be made equal to 1. This implies that

$$g_{jj}\sqrt{\sum_{i=1}^{n} d_{ij}^2} = 1 \tag{8}$$

where g_{jj} and d_{ij} are elements of the $\boldsymbol{G}$ and $\boldsymbol{D}$ matrices, respectively; hence, it follows that the jjth diagonal element of $\boldsymbol{G}$ is given by

$$g_{jj} = 1\Bigg/\sqrt{\sum_{i=1}^{n} d_{ij}^2} \qquad j = 1, 2, \ldots l \tag{9}$$

Conclusions

It is shown that matrices $\bar{\boldsymbol{D}}$ and $\bar{\boldsymbol{E}}$ can be brought to assume unique values by the *very simple* requirement that the length of each column of $\bar{\boldsymbol{D}}$ be made equal to 1. The numerical values of the product $\boldsymbol{D}\gamma\boldsymbol{E}$ term in Eq. (1) remain unchanged while yielding unique values for both the $\bar{\boldsymbol{D}}$ and $\bar{\boldsymbol{E}}$ matrices.

References

[1] Karpel, M., "Design for Active and Passive Flutter Suppression and Gust Alleviation," NASA CR-3482, 1981.

[2] Mor, M., and Livne, E., "State-Space Unsteady Aerodynamics for Aeroservoelastic Shape Optimization of Flight Vehicles," AIAA Paper 2004-1762, April 2004.